

Symmetries of the free Schrödinger equation in the non-commutative plane

Carles Batlle^a, Joaquim Gomis^b and Kiyoshi Kamimura^c

^a *Departament de Matemàtica Aplicada 4 and Institut d'Organització i Control, Universitat Politècnica de Catalunya - BarcelonaTech, EPSEVG, Av. V. Balaguer 1, 08800 Vilanova i la Geltrú, Spain*

^b *Departament d'Estructura i Constituents de la Matèria and Institut de Ciències del Cosmos, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain*

^c *Department of Physics, Toho University, Funabashi, Chiba 274-8510, Japan*

E-mails: carles.batlle@upc.edu, gomis@ecm.ub.es, kamimura@ph.sci.toho-u.ac.jp,

Abstract

We study the symmetries of the free Schrödinger equation in the non-commutative plane. These symmetry transformations form an infinite dimensional Weyl algebra that appears naturally from a two dimensional Heisenberg algebra generated by boosts and momenta. A finite dimensional subalgebra is the Schrödinger algebra which apart from the Galilei generators has dilatation and expansion. We consider the quantizations in both the reduced and extended phase spaces.

1 Introduction and results

The symmetries of a free massive non-relativistic particle and the associated Schrödinger equation have been investigated. The projective symmetries of the Schrödinger equation induced by the transformation on the coordinates (t, \vec{x}) are well known. They form the Schrödinger group [1] [2] [3] [4] that, apart from the Galilei symmetries, contains the dilatation and the expansion. Recently Valenzuela [5] (see also [6]) discussed higher-order symmetries of the free Schrödinger equation. These symmetry transformations form an infinite dimensional Weyl algebra constructed from the generators of space-translation and the ordinary commuting Galilean boost. The extra symmetries that do not belong to the Schrödinger group correspond to higher spin symmetries. These transformations are not induced by the transformations on the coordinates but they map solutions into solutions of the Schrödinger equation.

In the case of 2+1 dimensions, the Galilei group admits two central extensions [7, 8, 9], one associated to the exotic non-commuting boost and other appearing in the commutator of the ordinary boost and spatial translations. The non-relativistic particle in the non-commutative plane was introduced in [10] by considering a higher order Galilean invariant Lagrangian for the coordinates (t, \vec{x}) of the particle. A first order Lagrangian depending on the coordinates (t, \vec{x}) and extra coordinates \vec{v} was introduced in [11]. For these Lagrangians there are two possible realizations, one with non-commuting (exotic) boosts, and the order with ordinary commuting boosts [9]

In this paper we study all the classical symmetries of a massive free particle in the non-commutative plane. As we will see they are constructed from the Heisenberg algebra of commuting boost X_i and the generators of translations P_i , $\{X_i, P_j\} = \delta_{ij}$, $(i, j = 1, 2)$, all of which are constants of motion. The algebra of these symmetries is the infinite dimensional Weyl algebra associated with the Heisenberg algebra.

The subset of generators constructed up to quadratic polynomial of (X_i, P_j) form a finite dimensional subalgebra, which in turn contains the 9-dimensional Schrödinger algebra. A general element of the Weyl algebra is given by $\mathfrak{G}(X_i, P_j)$. We study the realization of these algebras in the classical unreduced phase-space, as well as in the reduced one, which appears due to the presence of second class constraints.

We have also studied all Noether symmetries of the free Schrödinger equation in the non-commutative plane, which also form a infinite dimensional Weyl algebra. The generators of the Schrödinger algebra are constructed in the quantum reduced phase space, as well as in the extended one. In the extended space we impose non-hermitean combinations of the second class constraints. In this case we consider two representations for the physical states, namely a Fock representation [12] and a coordinate representation. We show the equivalence between the reduced and extended space formulations.

The organization of the paper is as follows. In section 2 we study the classical symmetries of the massive particle in the non-commutative plane. Section 3 is devoted to study the quantum symmetries of the Schrödinger equation.

2 Classical symmetries of the non-relativistic particle Lagrangian in the non-commutative plane

The first order Lagrangian of a non-relativistic particle in the non-commutative plane, see for example [11], is given by, ($i, j = 1, 2$),

$$\mathcal{L}_{nc} = m \left(v_i \dot{x}_i - \frac{v_i^2}{2} \right) + \frac{\kappa}{2} \epsilon_{ij} v_i \dot{v}_j. \quad (2.1)$$

This Lagrangian can be obtained using the NLR method [13] applied to the exotic Galilei group in 2+1 dimensions¹, see [14] for the case of exotic Newton-Hooke whose flat limit gives (2.1). The coordinates x_i 's are the Goldstone bosons of the transverse translations and v_i 's are the Goldstone bosons of broken boost. The v_i 's and κ are dimensionless.

The Lagrangian (2.1) gives two primary second class constraints

$$\begin{aligned} \Pi_i &= \pi_i + \frac{\kappa}{2} \epsilon_{ij} v_j \approx 0, \\ V_i &= p_i - m v_i \approx 0, \end{aligned} \quad (2.2)$$

where p_i and π_i are the momenta canonically conjugate to x_i and v_i . The constraints (2.2) satisfy relations

$$\{\Pi_i, \Pi_j\} = \kappa \epsilon_{ij}, \quad \{V_i, V_j\} = 0, \quad \{\Pi_i, V_j\} = m \delta_{ij}, \quad (2.3)$$

and the Dirac hamiltonian is

$$H = \frac{p_i^2}{2m}, \quad (2.4)$$

up to quadratic terms in the constraints.

From the canonical pairs (x, v, p, π) we get a new set of canonical pairs $(\tilde{x}, \tilde{v}, \tilde{p}, \tilde{\pi})$ as

$$\begin{pmatrix} \tilde{x} \\ \tilde{p} \\ \tilde{v} \\ \tilde{\pi} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\kappa}{2m^2} \epsilon & \frac{\kappa}{2m} \epsilon & -\frac{1}{m} \\ & 1 & & \\ & -\frac{1}{m} & 1 & \\ & \frac{\kappa}{2m} \epsilon & & 1 \end{pmatrix} \begin{pmatrix} x \\ p \\ v \\ \pi \end{pmatrix}. \quad (2.5)$$

In terms of new variables the constraints (2.2) become a canonical pair,

$$\tilde{v}_i = -\frac{1}{m} V_i \approx 0, \quad \tilde{\pi}_i = \Pi_i + \frac{\kappa}{2m} \epsilon_{ij} V_j \approx 0. \quad (2.6)$$

The position and momentum of the particle are expressed as

$$x_i = \tilde{x}_i - \frac{\kappa}{2m^2} \epsilon_{ij} \tilde{p}_j - \frac{\kappa}{2m} \epsilon_{ij} \tilde{v}_j + \frac{1}{m} \tilde{\pi}_i, \quad p_i = \tilde{p}_i, \quad (2.7)$$

¹ Note that this Lagrangian is not dynamically equivalent to the higher order Lagrangian for a non-relativistic particle proposed in [10]. It can be obtained from (2.1) using the inverse Higgs mechanism [15].

and the Dirac Hamiltonian is (2.4) is written as

$$H = \frac{1}{2m} \tilde{p}_i^2. \quad (2.8)$$

The phase space is a direct product of two spaces. One is spanned by $(\tilde{v}, \tilde{\pi})$ with the constraints (2.6)

$$\tilde{v}_i \approx 0, \quad \tilde{\pi}_i \approx 0 \quad (2.9)$$

and thus classically trivial. The other one is spanned by (\tilde{x}, \tilde{p}) with the Hamiltonian (2.8). It is a system of free non-relativistic particle in 2D but with the coordinates \tilde{x}_i . We will see how the Schrödinger, or the more general symmetry algebras, are realized on it.

Classical Noether symmetries are generated by constants of motion which are arbitrary functions $\mathfrak{G}(X_i, P_j)$ of

$$X_i = \tilde{x}_i(0) = \tilde{x}_i(t) - \frac{t}{m} \tilde{p}_i(t), \quad \text{and} \quad P_i = \tilde{p}_i(0) = \tilde{p}_i(t), \quad (2.10)$$

verifying

$$\{P_i, P_j\} = 0, \quad \{X_i, P_j\} = \delta_{ij}, \quad \{X_i, X_j\} = 0. \quad (2.11)$$

The Lagrangian (2.1) is quasi-invariant under the transformation generated by $\mathfrak{G}(X_i, P_j)$. The canonical variation of (x, v) are

$$\begin{aligned} \delta x_i &= \frac{\partial \mathfrak{G}}{\partial p_i} = \frac{\partial \mathfrak{G}}{\partial P_i} - \frac{t}{m} \frac{\partial \mathfrak{G}}{\partial X_i} + \frac{\kappa}{2m^2} \epsilon_{ij} \frac{\partial \mathfrak{G}}{\partial X_j}, \\ \delta v_i &= \frac{\partial \mathfrak{G}}{\partial \pi_i} = -\frac{1}{m} \frac{\partial \mathfrak{G}}{\partial X_i}. \end{aligned} \quad (2.12)$$

When we consider the variation of the Lagrangian (2.1) $(\delta x, \delta v)$ are (2.12) in which (p_i, π_i) are replaced by, using the definition of momenta (2.2),

$$p_i \rightarrow mv_i, \quad \pi_i \rightarrow -\frac{\kappa}{2} \epsilon_{ij} v_j, \quad X_i \rightarrow x_i - tv_i + \frac{\kappa}{2m} \epsilon_{ij} v_j, \quad (2.13)$$

It follows that the variation of the Lagrangian becomes a total derivative,

$$\begin{aligned} \delta \mathcal{L}_{nc} &= \frac{d}{d\tau} \mathfrak{F}(x, v, t), \\ \mathfrak{F}(x, v, t) &= [p_i \delta x_i + \pi_i \delta v_i - \mathfrak{G}]_{p_i=mv_i, \pi_i=-\frac{\kappa}{2} \epsilon_{ij} v_j} \\ &= \left[mv_i \left(\frac{\partial \mathfrak{G}}{\partial P_i} - \frac{t}{m} \frac{\partial \mathfrak{G}}{\partial X_i} \right) - \mathfrak{G} \right]_{p_i=mv_i, \pi_i=-\frac{\kappa}{2} \epsilon_{ij} v_j}. \end{aligned} \quad (2.14)$$

2.1 Galilean symmetries

We start by considering the Galilean symmetries of (2.1), the action is invariant under translations and boost

$$x'_i = x_i + \alpha_i \quad v'_i = v_i \quad (2.15)$$

$$x'_i = x_i - \beta_i t, \quad v'_i = v_i - \beta_i \quad (2.16)$$

and under rotations and time translations,

$$x'_i = x_i \cos \varphi + \epsilon_{ij} x_j \sin \varphi, \quad v'_i = v_i \cos \varphi + \epsilon_{ij} v_j \sin \varphi, \quad (2.17)$$

$$t' = t - \gamma. \quad (2.18)$$

The corresponding Noether charges of translations and boosts are given as

$$P_i = p_i, \quad K_i = mx_i - p_i t - \pi_i + \frac{\kappa}{2} \epsilon_{ij} v_j = mX_i + \frac{\kappa}{2m} \epsilon_{ij} P_j, \quad (2.19)$$

while the angular momentum is

$$J = \epsilon_{ij} (x_i p_j + v_i \pi_j) = \epsilon_{ij} (X_i P_j + \tilde{v}_i \tilde{\pi}_j). \quad (2.20)$$

Together with total Hamiltonian (2.4). They generate the exotic Galilei algebra [7, 8, 9]

$$\{H, J\} = 0, \quad (2.21)$$

$$\{H, K_i\} = -P_i, \quad \{H, P_i\} = 0, \quad (2.22)$$

$$\{J, P_i\} = \epsilon_{ij} P_j, \quad \{J, K_i\} = \epsilon_{ij} K_j, \quad (2.23)$$

$$\{K_i, P_j\} = m \delta_{ij}, \quad \{K_i, K_j\} = -\kappa \epsilon_{ij}, \quad (2.24)$$

$$\{P_i, P_j\} = 0. \quad (2.25)$$

It seems that the Lagrangian (2.1) gives a phase space realization of the 2+1 Galilei group with two central charges m, κ . However one of the central charges is trivial, since, if we modify the generator of boost as in [9],

$$\tilde{K}_i = K_i - \frac{\kappa}{2m} \epsilon_{ij} P_j = mX_i = mx_i - \pi_i + \frac{1}{2} \kappa \epsilon_{ij} v_j - \frac{\kappa}{2m} \epsilon_{ij} p_j - p_i t \quad (2.26)$$

(H, P, \tilde{K}, J) verifies standard Galilean algebra without κ . The physical change of the boost generators is to shift the parameter of translations

$$\alpha_i \rightarrow \alpha_i + \frac{\kappa}{2m} \epsilon_{ij} \beta_j. \quad (2.27)$$

Note that modified boost generators \tilde{K}_i are proportional to the coordinates at $t = 0$, $X_i = \tilde{x}^i(0)$, that verify $\{X_i, X_j\} = 0$ and we have a realization with only one non-trivial central charge associated to the mass of the particle².

² Note however that $\delta_{K_i} L = \delta_{\tilde{K}_i} L = \frac{d}{dt} (-mx_i - \frac{\kappa}{2} \epsilon_{ij} v_j) \beta_i$, where β_i is boost parameter.

2.2 Schrödinger symmetries

Note that X_i, P_j in (2.11) form a two dimensional Heisenberg algebra

$$\mathfrak{h}_2 = \{1, X_i, P_i ; \quad \{X_i, P_j\} = \delta_{ij}, \quad i, j = 1, 2\}.$$

The Weyl algebra, denoted by $[\mathfrak{h}_2^*]$, can be defined as one generated by (the Weyl ordered) polynomials of the Heisenberg algebra generators, (X_i, P_i) . $[\mathfrak{h}_2^*]$ is the infinite higher dimensional algebra of a particle in the non-commutative plane. There are finite dimensional subalgebras of the higher spin algebra whose generators are constructed from the product of generators X_i, P_j up to second order [5];

$$\mathfrak{h}_2 \subset Galilei \subset Sch(2) \subset \mathfrak{h}_2 \oplus \mathfrak{sp}(4) \subset [\mathfrak{h}_2^*]. \quad (2.28)$$

$Sch(2)$ is the Schrödinger algebra in 2D, whose generators are those of the Galilean algebra X_i, P_i, H, J , and the dilatation, D , and the expansion, C , given by

$$D = X_i P_i = x_i p_i - \frac{t}{m} p_i^2 - \frac{1}{m} \pi_i p_i + \frac{\kappa}{2m} \epsilon_{ij} p_i v_j, \quad (2.29)$$

$$\begin{aligned} C = m X_i X_i &= m x_i^2 + \frac{1}{m} t^2 p_i^2 + \frac{1}{m} \pi_i^2 + \frac{\kappa^2}{4m} v_i^2 + \frac{\kappa^2}{4m^3} p_i^2 \\ &- 2t x_i p_i - 2x_i \pi_i + \kappa \epsilon_{ij} x_i v_j - \frac{\kappa}{m} \epsilon_{ij} x_i p_j + \frac{2}{m} t p_i \pi_i \\ &- \frac{\kappa}{m} t \epsilon_{ij} p_i v_j - \frac{\kappa}{m} \epsilon_{ij} \pi_i v_j + \frac{\kappa}{m^2} \epsilon_{ij} \pi_i p_j - \frac{\kappa^2}{2m^2} v_i p_i. \end{aligned} \quad (2.30)$$

In the same spirit, we also redefine the generator of rotations as

$$J = \epsilon_{ij} X_i P_j = \epsilon_{ij} x_i p_j - \frac{\kappa}{2m^2} p_i^2 + \frac{\kappa}{2m} v_i p_i + \frac{1}{m} \epsilon_{ij} p_i \pi_j \quad (2.31)$$

which, up to square of constraints, coincides with (2.20).

The new, non-zero Poisson brackets are

$$\begin{aligned} \{D, C\} &= -2C, & \{D, H\} &= 2H, & \{H, C\} &= -2D, \\ \{D, P_i\} &= P_i, & \{D, X_i\} &= -X_i, & \{C, P_i\} &= 2m X_i. \end{aligned} \quad (2.32)$$

The transformations of the coordinates x_i, v_i under dilatation and expansion are obtained from (2.12) as

$$\delta_D x_i = \frac{\alpha}{m} (m x_i - 2m t v_i + \kappa \epsilon_{ij} v_j), \quad (2.33)$$

$$\delta_D v_i = -\alpha v_i,$$

$$\delta_C x_i = \frac{\lambda}{m} (2m t^2 v_i - 2m t x_i + \kappa \epsilon_{ij} x_j - 2\kappa t \epsilon_{ij} v_j - \frac{\kappa^2}{2m} v_i), \quad (2.34)$$

$$\delta_C v_i = \frac{\lambda}{m} (-2m x_i + 2m t v_i - \kappa \epsilon_{ij} v_j), \quad (2.35)$$

where α and λ are the corresponding infinitesimal parameters.

2.3 Reduction of second class constraints

The classical symmetry algebra is also realized in the reduced phase space defined by the second class constraints $\Pi_i = V_i = 0$. The Dirac bracket is

$$\{A, B\}^* = \{A, B\} + \{A, \Pi_i\} \frac{1}{m} \{V_i, B\} - \{A, V_i\} \frac{1}{m} \{\Pi_i, B\} - \{A, V_i\} \frac{\kappa \epsilon_{ij}}{m^2} \{V_j, B\} \quad (2.36)$$

and yields

$$\{x_i, x_j\}^* = \frac{\kappa}{m^2} \epsilon_{ij}, \quad \{x_i, p_j\}^* = \delta_{ij}, \quad \{p_i, p_j\}^* = 0. \quad (2.37)$$

In this space, the symmetry transformations are generated using the Dirac bracket and the reduced generators, which can be obtained by substituting $v_i = p_i/m$, $\pi_i = -\kappa/(2m)\epsilon_{ij}p_j$ into the standard ones. In particular the Schrödinger generators are given by

$$P_i^{(R)} = p_i, \quad (2.38)$$

$$K_i^{(R)} = mx_i - tp_i + \frac{\kappa}{m} \epsilon_{ij} p_j, \quad (\text{exotic Gal})$$

$$\tilde{K}_i^{(R)} = K_i^{(R)} - \frac{\kappa}{2m} \epsilon_{ij} P_j^{(R)} = mx_i - tp_i + \frac{\kappa}{2m} \epsilon_{ij} p_j, \quad (\text{standard Gal}) \quad (2.39)$$

$$H^{(R)} = \frac{1}{2m} p_i^2, \quad (2.40)$$

$$J^{(R)} = \epsilon_{ij} x_i p_j + \frac{\kappa}{2m^2} p_i^2, \quad (2.41)$$

$$D^{(R)} = p_i x_i - \frac{1}{m} t p_i^2, \quad (2.42)$$

$$C^{(R)} = mx_i^2 + \frac{1}{m} t^2 p_i^2 + \frac{\kappa^2}{4m^3} p_i^2 - 2tx_i p_i + \frac{\kappa}{m} \epsilon_{ij} x_i p_j. \quad (2.43)$$

They generate the Schrödinger algebra with the Dirac bracket, since $\tilde{K}_i^{(R)}$, $P_i^{(R)}$ generate a Heisenberg algebra:

$$\left\{ \tilde{K}_i^{(R)}, P_j^{(R)} \right\}^* = m \delta_{ij}, \quad \left\{ P_i^{(R)}, P_j^{(R)} \right\}^* = 0, \quad \text{and} \quad \left\{ \tilde{K}_i^{(R)}, \tilde{K}_j^{(R)} \right\}^* = 0. \quad (2.44)$$

Symmetry transformations are generated either using the Poisson brackets in the original phase space or using the Dirac brackets with the reduced generators, (2.38)-(2.43). For example the "exotic Galilei" generators K_i are satisfying

$$\{K_i, K_j\} = \left\{ K_i^{(R)}, K_j^{(R)} \right\}^* = -\kappa \epsilon_{ij}, \quad (2.45)$$

and generate "standard(covariant) Galilei" transformation of (x_i, p_i) as

$$\begin{aligned} \delta x_i &= \{x_i, \beta \cdot K\} = \{x_i, \beta \cdot K^{(R)}\}^* = -t \beta_i, \\ \delta p_i &= \{p_i, \beta \cdot K\} = \{p_i, \beta \cdot K^{(R)}\}^* = -m \beta_i. \end{aligned} \quad (2.46)$$

The "standard Galilei" generators \tilde{K}_i are satisfying

$$\left\{ \tilde{K}_i, \tilde{K}_j \right\} = \left\{ \tilde{K}_i^{(R)}, \tilde{K}_j^{(R)} \right\}^* = 0. \quad (2.47)$$

and generate "exotic(non-covariant) Galilei" transformation of (x_i, p_i)

$$\begin{aligned}\delta x_i &= \{x_i, \beta \cdot \tilde{K}\} = \{x_i, \beta \cdot \tilde{K}^{(R)}\}^* = -t\beta_i + \frac{\kappa}{2m}\epsilon_{ij}\beta_j, \\ \delta p_i &= \{p_i, \beta \cdot \tilde{K}\} = \{p_i, \beta \cdot \tilde{K}^{(R)}\}^* = -m\beta_i.\end{aligned}\tag{2.48}$$

3 Quantum symmetries of free Schrödinger equation in the non-commutative plane

In this section we will study the quantization of the model at the level of the Schrödinger equation and their symmetries. We will quantize it in two approaches, one in the reduced phase space and the other in the extended phase space.

3.1 Quantization in the reduced phase space

In the classical theory x_i has a nonzero Dirac bracket $\{x_i, x_j\}^*$ as in (2.37) in the reduced phase space. Since Dirac brackets are replaced by commutators in the canonical quantization one cannot have a x_i -coordinate representation of quantum states³. To discuss symmetries of Schrödinger equations we introduce new coordinates

$$y_i \equiv x_i + \frac{\kappa}{2m^2}\epsilon_{ij}p_j, \quad q_i = p_i, \tag{3.1}$$

such that

$$\{y_i, y_j\}^* = 0, \quad \{y_i, q_j\}^* = \delta_{ij}, \quad \{q_i, q_j\}^* = 0. \tag{3.2}$$

The Schrödinger equation $(i\partial_t - H)|\Psi(t)\rangle = 0$ takes a form of free particle for the wave function

$$\Psi(y, t) = \langle y | \Psi(t) \rangle, \quad \hat{y}_i |y\rangle = y_i |y\rangle, \quad \langle y | y' \rangle = \delta^2(y - y'), \tag{3.3}$$

as

$$(i\partial_t - \frac{1}{2m}(-i\partial_y)^2)\Psi(y, t) = 0, \tag{3.4}$$

and the inner product is

$$\langle \Psi | \Psi \rangle = \int dy \overline{\Psi(y, t)} \Psi(y, t). \tag{3.5}$$

Note that y_i are not covariant under exotic Galilei transformation generated by K_i

$$\delta y_i = \{y_i, \beta \cdot K\} = \{y_i, \beta \cdot K^{(R)}\}^* = -\beta_i t - \frac{\kappa}{2m}\epsilon_{ij}\beta_j, \tag{3.6}$$

but covariant under the Galilei transformation generated by \tilde{K}_i

$$\delta y_i = \{y_i, \beta \cdot \tilde{K}\} = \{y_i, \beta \cdot \tilde{K}^{(R)}\}^* = -\beta_i t. \tag{3.7}$$

³Since q_i 's are commuting the momentum representation is possible[11].

The position operators, covariant under K_i , are

$$\hat{x}_i = y_i - \frac{\kappa}{2m^2} \epsilon_{ij} (-i\partial_{y_j}). \quad (3.8)$$

They are Hermitian since $\hat{y}_i = y_i$, $\hat{q}_i = -i\partial_{y_i}$ are Hermitian in appropriate boundary conditions on $\Psi(y, t)$.

Although in our free theory we are able to work with the operators \hat{y}_i , \hat{q}_i , the non-commutative position operator $\hat{x}_i = y_i - \frac{\kappa}{2m^2} \epsilon_{ij} (-i\partial_{y_j})$ may be necessary to describe interactions, for example with the electromagnetic field, which introduce couplings with a source at position x_i , see for example [11] [16] [17].

If we denote generically by $\mathfrak{G}^{(R)}(t, x, p) = \mathfrak{G}(X, P)|_{\Pi=V=0}$ the generators of the Weyl algebra in the reduced classical space the generators in this quantization are given by

$$\hat{\mathfrak{G}}_i^{(1)}(t, y, \hat{q}) = \mathfrak{G}_i^{(R)} \Big|_{x_j=y_j-\frac{\kappa}{2m^2}\epsilon_{jl}\hat{q}_l, p_j=\hat{q}_j} = \mathfrak{G}_i(y - \frac{t}{m}\hat{q}, \hat{q}), \quad (3.9)$$

with $\hat{q}_i = -i\partial/\partial y_i$ and with the appropriate dealing of operator ordering. In particular the Schrödinger generators are

$$\hat{P}_i^{(1)} = \hat{q}_i = -i\frac{\partial}{\partial y_i}, \quad (3.10)$$

$$\hat{K}_i^{(1)} = my_i - t\hat{q}_i = my_i + it\frac{\partial}{\partial y_i}, \quad (3.11)$$

$$\hat{H}^{(1)} = \frac{1}{2m}\hat{q}_i^2 = -\frac{1}{2m}\frac{\partial^2}{\partial y_i^2}, \quad (3.12)$$

$$\hat{J}^{(1)} = \epsilon_{ij}y_i\hat{q}_j = -i\epsilon_{ij}y_i\frac{\partial}{\partial y_j}, \quad (3.13)$$

$$\hat{D}^{(1)} = y_i\hat{q}_i - i - \frac{1}{m}t\hat{q}_i^2 = -iy_i\frac{\partial}{\partial y_i} + \frac{1}{m}t\frac{\partial^2}{\partial y_i^2} - i, \quad (3.14)$$

$$\hat{C}^{(1)} = my_i^2 - 2ty_i\hat{q}_i + 2it + \frac{1}{m}t^2\hat{q}_i^2 = my_i^2 + 2ity_i\frac{\partial}{\partial y_i} - \frac{1}{m}t^2\frac{\partial^2}{\partial y_i^2} + 2it, \quad (3.15)$$

where a Weyl ordering has been used for $\hat{D}^{(1)}$ and $\hat{C}^{(1)}$. These generators are Hermitian operators when acting on the wave functions $\Psi(t, y)$. Furthermore, they obey the abstract quantum Schrödinger algebra *off shell*, with non-zero commutators given by

$$\begin{aligned} [\hat{K}_i, \hat{P}_j] &= im\delta_{ij}, \quad [\hat{J}, \hat{P}_i] = i\epsilon_{ij}\hat{P}_j, \quad [\hat{J}, \hat{K}_i] = i\epsilon_{ij}\hat{K}_j, \quad [\hat{H}, \hat{K}_i] = -i\hat{P}_i, \\ [\hat{D}, \hat{H}] &= 2i\hat{H}, \quad [\hat{D}, \hat{P}_i] = i\hat{P}_i, \quad [\hat{D}, \hat{K}_i] = -i\hat{K}_i, \\ [\hat{D}, \hat{C}] &= -2i\hat{C}, \quad [\hat{H}, \hat{C}] = -2i\hat{D}, \quad [\hat{C}, \hat{P}_i] = 2i\hat{K}_i. \end{aligned} \quad (3.16)$$

Using these, together with

$$[i\partial_t, \hat{K}_i^{(1)}] = -i\hat{P}_i^{(1)}, \quad [i\partial_t, \hat{D}^{(1)}] = -2i\hat{H}^{(1)}, \quad [i\partial_t, \hat{C}^{(1)}] = -2i\hat{D}^{(1)}, \quad (3.17)$$

one can show that

$$\left[i\partial_t - \hat{H}^{(1)}, \hat{\mathfrak{G}}_i^{(1)} \right] = 0 \quad (3.18)$$

for all the generators $\hat{\mathfrak{G}}_i^{(1)}$, which proves the invariance of the Schrödinger equation under the Schrödinger transformations in this reduced space quantization.

The wave functions transform as

$$\Psi'(y, t) = e^{i\alpha_i \hat{\mathfrak{G}}_i^{(1)}(t, y, (-i\partial_y))} \Psi(y, t), \quad (3.19)$$

where α_i are the parameters of the transformations.

The *on-shell* Schrödinger transformations on the wave functions $\Psi(y, t)$ induce

$$\Psi'(y, t) = e^{A+iB} \Psi(y', t'); \quad (3.20)$$

where the coordinate transformations of (y, t) is the (N=1) conformal Galilean transformation and in the multiplicative factor e^{A+iB} , A and B are real. For each Schrödinger transformation we have (see, for instance, [3], [18])

1. H (time translation),

$$t' = t + a, \quad y' = y, \quad A = B = 0, \quad (3.21)$$

2. D (dilatation),

$$t' = e^\lambda t, \quad y' = e^{\frac{\lambda}{2}} y, \quad A = \frac{\lambda}{2}, \quad B = 0, \quad (3.22)$$

3. C (expansion),

$$t' = \frac{t}{1 - \kappa t}, \quad y'_i = \frac{y_i}{1 - \kappa t}, \quad e^A = \frac{1}{(1 - \kappa t)}, \quad B = -\frac{\kappa m y^2}{2(1 - \kappa t)}, \quad (3.23)$$

4. $\beta_i^0 P_i + \beta_i^1 X_i$, ($[\beta_i^0] = L$, $[\beta_i^1] = L^{-1}$), (spatial translations and boost)

$$t' = t, \quad y'_i = y_i + (\beta^0 + t \frac{\beta^1}{m})_i, \quad A = 0, \quad B = -2\pi\omega_1 = -m(y_i + \frac{1}{2}(\beta_i^0 + t \frac{\beta_i^1}{m})) \frac{\beta_i^1}{m}. \quad (3.24)$$

The difference of the transformation from one of ordinary Schrödinger equation is that in the non-commutative case the coordinates that are transformed by conformal Galilean transformations are the canonical one y_i and not in the physical positions of the particle x_i .

The invariance of the solutions of the Schrödinger equation under a general element of the Weyl algebra can be proved using the invariance under the generators of the Heisenberg algebra and commutator properties (3.17).

3.2 Quantization in the extended phase space

In order to quantize the model in the extended phase space the second class constraints (2.2) are imposed as physical state conditions by taking their non-hermitean combinations as in [14]. We first consider the canonical transformation (2.5) that separates the second class constraints as new coordinates. It is realized at quantum level as a unitary transformation

$$\tilde{q} = U^\dagger q U, \quad U = e^{\frac{i}{m} p_i (\pi_i - \frac{\kappa}{2} \epsilon_{ij} v_j)}, \quad (3.25)$$

For example,

$$\tilde{x}_i = U^\dagger x_i U = x_i - \frac{1}{m} (\pi_i - \frac{\kappa}{2} \epsilon_{ij} v_j) + \frac{1}{2} \frac{\kappa}{m} \epsilon_{ij} (-\frac{p_j}{m}). \quad (3.26)$$

It is useful to introduce the complex combinations of the phase space variables $\tilde{\pi}_\pm = \tilde{\pi}_1 \pm i\tilde{\pi}_2$ and $\tilde{v}_\pm = \tilde{v}_1 \pm i\tilde{v}_2$, which allow us to introduce two pairs of annihilation and creation operators

$$\tilde{a}_\pm = \frac{i}{\sqrt{2\kappa}} (\tilde{\pi}_\pm - i\frac{\kappa}{2} \tilde{v}_\pm), \quad \tilde{a}_\pm^\dagger = \frac{-i}{\sqrt{2\kappa}} (\tilde{\pi}_\mp + i\frac{\kappa}{2} \tilde{v}_\mp). \quad (3.27)$$

The creation operators \tilde{a}_\pm^\dagger are their hermitean conjugate satisfying

$$[\tilde{a}_\pm, \tilde{a}_\pm^\dagger] = 1, \quad \text{others} = 0. \quad (3.28)$$

Using the Fock representation for $(\tilde{v}, \tilde{\pi})$ and coordinate representation for (\tilde{x}, \tilde{p}) , any state of this system is described by

$$|\Psi(t)\rangle = \sum_{n_+ \geq 0, n_- \geq 0} \int dy |n_+, n_-\rangle \otimes |y\rangle \Phi_{n_+ n_-}(y, t), \quad (3.29)$$

where $|n_+, n_-\rangle$ is the eigenstate of $\tilde{N}_\pm = \tilde{a}_\pm^\dagger \tilde{a}_\pm$ with eigenvalues $n_\pm \in \mathbb{N} \cup \{0\}$ and $|y\rangle$ is the eigenstate of commuting operators \tilde{x}_i with eigenvalue y_i . They are normalized as

$$\langle n_+, n_- | n'_+, n'_- \rangle = \delta_{n_+ n'_+} \delta_{n_- n'_-}, \quad \langle y | y' \rangle = \delta^2(y - y'). \quad (3.30)$$

The scalar product is given by

$$\langle \Psi | \Psi' \rangle = \sum_{n_\pm} \int dy \overline{\Phi_{n_+ n_-}(y, t)} \Phi'_{n_+ n_-}(y, t). \quad (3.31)$$

In the quantization in the extended phase space the second class constraints (2.2) are imposed as the physical state conditions by taking their non-hermitean combination,

$$\tilde{a}_\pm |\Psi_{phys}(t)\rangle = 0. \quad (3.32)$$

It means physical states are minimum uncertainty states in $(\tilde{v}, \tilde{\pi})$. It selects out only $n_+ = n_- = 0$ state and $\Phi_{n_+n_-}(y, t) = 0$ except for $\Phi_{0,0}(y, t) \equiv \Phi_0(y, t)$,

$$|\Psi_{phys}(t)\rangle = \int dy |0, 0\rangle \otimes |y\rangle \Phi_0(y, t). \quad (3.33)$$

The Schrödinger equation is

$$(i\partial_t - H)|\Psi_{phys}(t)\rangle = 0, \quad H = \frac{\hat{p}^2}{2m}, \quad (3.34)$$

and thus

$$(i\partial_t - H)\Phi_0(y, t) = 0, \quad H = \frac{1}{2m}(-i\partial_{y_i})^2. \quad (3.35)$$

Generators of Schrödinger algebra \mathfrak{S}_i in the extended space are written in bilinear forms of

$$X_i = \tilde{x}_i(0) = \tilde{x}_i(t) - t\tilde{p}_i(t), \quad P_i = \tilde{p}_i(0) = \tilde{p}_i(t), \quad (3.36)$$

in (2.10)⁴. Since they commute with \tilde{a}_\pm and \tilde{a}_\pm^\dagger physical states remain invariant. They act on the physical states as

$$|\Psi_{phys}(t)\rangle \rightarrow |\Psi'_{phys}(t)\rangle = e^{i\mathfrak{S}(X,P)}|\Psi_{phys}(t)\rangle \quad (3.37)$$

it turns out the transformation of the wave function $\Phi_0(y, t)$

$$\Phi'_0(y, t) = e^{i\mathfrak{S}(X,P)}\Phi_0(y, t) = e^{i\mathfrak{S}(y-t(-i\partial_y), (-i\partial_y))}\Phi_0(y, t). \quad (3.38)$$

This transformation has the same form as one in the reduced phase space generated by (3.9) -(3.15). Then the wave function in the reduced space $\Psi(y, t) = \langle y|\Psi(t)\rangle$ and $\Phi_0(y, t) = \langle y|\otimes\langle 00|\Psi(t)\rangle$ that appear in the extended space quantization are identified. Note in the former $\langle y|$ is eigenstate of $\hat{y}_i = x_i + \frac{\kappa}{2m^2}\epsilon_{ij}p_j$ in (3.1) but $\langle y|$ in the latter is eigenstate of \hat{x}_i that are commuting in the extended space.

We can see how the non-commutativity of the position operators appears. $\hat{x}_\pm = x_1 \pm ix_2$ are commuting in the extended phase space. Using (2.5) we write

$$\begin{aligned} x_+ &= \tilde{x}_+ + i\frac{\kappa}{2m^2}\tilde{p}_+ + i\sqrt{\frac{2\kappa}{m^2}}\tilde{a}_-^\dagger, \\ x_- &= \tilde{x}_- - i\frac{\kappa}{2m^2}\tilde{p}_- - i\sqrt{\frac{2\kappa}{m^2}}\tilde{a}_- = x_+^\dagger. \end{aligned} \quad (3.39)$$

In the reduced space quantization \tilde{a}_\pm are effectively put to zero and x_\pm becomes non-commutative operator on $|\Psi(t)\rangle$. On the other hand in the quantization in the extended space expectation values of the position operators between two physical states are

$$\begin{aligned} \langle \Psi|\hat{x}_\pm|\Psi' \rangle &= \int dy dy' \overline{\Phi_0(y, t)} \langle y|\langle 0|(\tilde{x}_\pm \pm i\frac{\kappa}{2m^2}\tilde{p}_\pm \pm i\sqrt{\frac{2\kappa}{m^2}}\begin{pmatrix} \tilde{a}_-^\dagger \\ \tilde{a}_- \end{pmatrix})|0\rangle|y'\rangle \Phi'_0(y', t) \\ &= \int dy \overline{\Phi_0(y, t)}(y_\pm \pm i\frac{\kappa}{2m^2}(-2i\partial_{y_\pm}))\Phi'_0(y, t). \end{aligned} \quad (3.40)$$

⁴Although the angular momentum J in (2.20) contains a term that rotates constrained sector, which is trivially invariant under rotation.

Commutative position operators \hat{x}_\pm on states $|\Psi\rangle$ act as non-commutative operators $(y_\pm \pm i\frac{\kappa}{2m^2}(-2i\partial_{y_\pm}))$ on the wave functions $\Phi_0(y, t)$.

It is useful to consider the unitary transformation U on the creation and annihilation operators $\tilde{a}_\pm, \tilde{a}_\pm^\dagger$,

$$\begin{aligned}\tilde{a}_+ &= U^\dagger a_+ U = a_+, \\ \tilde{a}_- &= U^\dagger a_- U = a_- - \sqrt{\frac{\kappa}{2m^2}} p_-.\end{aligned}\tag{3.41}$$

The quantization in the extended phase space can be also done by considering the constraint equations (3.32) in terms of the operators a_\pm, a_\pm^\dagger . The physical state conditions (3.32) are

$$a_+ |\Psi_{phys}(t)\rangle = 0, \quad (p_- - \sqrt{\frac{2m^2}{\kappa}} a_-) |\Psi_{phys}(t)\rangle = 0.\tag{3.42}$$

It is a coherent state of a_- with eigenvalue $\sqrt{\frac{\kappa}{2m^2}} p_-$ [12]. The Schrödinger generators in this representation are

$$\begin{aligned}X_\pm^{(2)} &= (x_\pm \mp i\frac{\kappa}{2m^2} p_\pm) - \frac{t}{m} p_\pm \pm i\frac{\kappa}{m^2} (p_\pm - \sqrt{\frac{2m^2}{\kappa}} \begin{pmatrix} a_-^\dagger \\ a_- \end{pmatrix}), \\ P_\pm^{(2)} &= p_\pm = -2i\partial_{x_\mp}, \quad [x_\pm, p_\mp] = 2i, \\ D^{(2)} &= \frac{1}{2} \left((x_+ p_- + p_+ x_- - \frac{2t}{m} p_+ p_-) + i\frac{\kappa}{m^2} (p_+ - \sqrt{\frac{2m^2}{\kappa}} a_-^\dagger) p_- - i\frac{\kappa}{m^2} p_+ (p_- - \sqrt{\frac{2m^2}{\kappa}} a_-) \right), \\ C^{(2)} &= \frac{1}{2} \left((x_+ - i\frac{\kappa}{2m^2} p_+) (x_- + i\frac{\kappa}{2m^2} p_-) - \frac{t}{m} ((x_+ - i\frac{\kappa}{2m^2} p_+) p_- + p_+ (x_- + i\frac{\kappa}{2m^2} p_-)) \right. \\ &\quad + \frac{t^2}{2m^2} p_+ p_- + \frac{1}{2} ((x_+ - i\frac{\kappa}{2m^2} p_+) - \frac{t}{m} p_+) (-i\frac{\kappa}{m^2}) (p_- - \sqrt{\frac{2m^2}{\kappa}} a_-) \\ &\quad + \frac{1}{2} i\frac{\kappa}{m^2} (p_+ - \sqrt{\frac{2m^2}{\kappa}} a_-^\dagger) ((x_- + i\frac{\kappa}{2m^2} p_-) - \frac{t}{m} p_-) \\ &\quad \left. + \frac{1}{2} (i\frac{\kappa}{m^2} (p_+ - \sqrt{\frac{2m^2}{\kappa}} a_-^\dagger)) (-i\frac{\kappa}{m^2} (p_- - \sqrt{\frac{2m^2}{\kappa}} a_-)) \right), \\ J^{(2)} &= \frac{i}{2} \left((x_+ p_- - p_+ x_- - i\frac{\kappa}{m^2} p_+ p_-) + i\frac{\kappa}{m^2} (p_+ - \sqrt{\frac{2m^2}{\kappa}} a_-^\dagger) p_- + i\frac{\kappa}{m^2} p_+ (p_- - \sqrt{\frac{2m^2}{\kappa}} a_-) \right).\end{aligned}\tag{3.43}$$

These generators commute with the constraint equations and the Schrödinger operator $(i\partial_t - H)$. Notice that the set of generators do not depend on a_+, a_+^\dagger therefore the transition to the Fock space used in [12] is recovered.

The Fock expression of a generic element of the Weyl algebra $\mathfrak{G}(X, P)$ can be obtained using the expression of the operators X and P given by (2.10).

3.2.1 Coordinate representation

In the representation of coordinates the time Schrödinger equation and the constraint equations (3.41) in the non-commutative plane becomes [14]

$$\hat{S}_1 \Psi \equiv \left(\frac{\partial}{\partial v_-} + \frac{\kappa}{4} v_+ \right) \Psi(x, v, t) = 0, \quad (3.44)$$

$$\hat{S}_2 \Psi \equiv \left(\frac{\partial}{\partial x_+} - i \frac{m}{4} v_- - i \frac{m}{\kappa} \frac{\partial}{\partial v_+} \right) \Psi(x, v, t) = 0, \quad (3.45)$$

$$\hat{S}_3 \Psi \equiv \left(i \frac{\partial}{\partial t} + \frac{2}{m} \frac{\partial^2}{\partial x_+ \partial x_-} \right) \Psi(x, v, t) = 0. \quad (3.46)$$

In this representation, the operators associated to the generators of the Heisenberg algebra are

$$\begin{aligned} \hat{P}_1 &= -i \frac{\partial}{\partial x_+} - i \frac{\partial}{\partial x_-}, \\ \hat{P}_2 &= \frac{\partial}{\partial x_+} - \frac{\partial}{\partial x_-}, \\ \hat{K}_1 &= \frac{m}{2} (x_+ + x_-) + \left(it - \frac{\kappa}{2m} \right) \frac{\partial}{\partial x_+} + \left(it + \frac{\kappa}{2m} \right) \frac{\partial}{\partial x_-} + \frac{\kappa}{4i} (v_+ - v_-) + i \frac{\partial}{\partial v_+} + i \frac{\partial}{\partial v_-}, \\ \hat{K}_2 &= \frac{m}{2i} (x_+ - x_-) - \left(t + i \frac{\kappa}{2m} \right) \frac{\partial}{\partial x_+} + \left(t - i \frac{\kappa}{2m} \right) \frac{\partial}{\partial x_-} - \frac{\kappa}{4} (v_+ + v_-) - \frac{\partial}{\partial v_+} + \frac{\partial}{\partial v_-}, \end{aligned} \quad (3.47)$$

or, in covariant form,

$$\hat{P}_i = -i \frac{\partial}{\partial x_i}, \quad (3.48)$$

$$\hat{K}_i = m x_i + it \frac{\partial}{\partial x_i} + i \frac{\kappa}{2m} \epsilon_{ij} \frac{\partial}{\partial x_j} + \frac{\kappa}{2} \epsilon_{ij} v_j + i \frac{\partial}{\partial v_i}, \quad (3.49)$$

which, indeed, satisfy $[\hat{P}_i, \hat{K}_j] = -im \delta_{ij}$, with all the other commutators equal to zero.

It is immediate to check that the operators \hat{P}_i , \hat{K}_i commute with all of \hat{S}_1 , \hat{S}_2 and \hat{S}_3 , and hence that they generate Schrödinger symmetries for the free particle in the non-commutative plane.

The rest of generators of the Schrödinger algebra are given by

$$\hat{H} = -\frac{2}{m} \frac{\partial^2}{\partial x_+ \partial x_-} = -\frac{1}{2m} \frac{\partial^2}{\partial x_i^2}, \quad (3.50)$$

$$\hat{J} = -i \epsilon_{ij} x_i \frac{\partial}{\partial x_j} + \frac{\kappa}{2m^2} \frac{\partial^2}{\partial x_i^2} - i \frac{\kappa}{2m} v_i \frac{\partial}{\partial x_i} - \frac{1}{m} \epsilon_{ij} \frac{\partial^2}{\partial x_i \partial v_j}, \quad (3.51)$$

$$\hat{D} = -i x_i \frac{\partial}{\partial x_i} + \frac{1}{m} t \frac{\partial^2}{\partial x_i^2} + \frac{1}{m} \frac{\partial^2}{\partial x_i \partial v_i} + i \frac{\kappa}{2m} \epsilon_{ij} v_i \frac{\partial}{\partial x_j} - i, \quad (3.52)$$

$$\begin{aligned}
\hat{C} = & 2itx_i \frac{\partial}{\partial x_i} + i \frac{\kappa^2}{2m^2} v_i \frac{\partial}{\partial x_i} + i \frac{\kappa}{m} \epsilon_{ij} x_i \frac{\partial}{\partial x_j} - i \frac{\kappa}{m} t \epsilon_{ij} v_i \frac{\partial}{\partial x_j} \\
& - i \frac{\kappa}{m} \epsilon_{ij} v_i \frac{\partial}{\partial v_j} + 2ix_i \frac{\partial}{\partial v_i} - \frac{1}{m} t^2 \frac{\partial^2}{\partial x_i^2} - \frac{\kappa^2}{4m^3} \frac{\partial^2}{\partial x_i^2} - \frac{1}{m} \frac{\partial^2}{\partial v_i^2} \\
& - \frac{2}{m} t \frac{\partial^2}{\partial x_i \partial v_i} + \frac{\kappa}{m^2} \epsilon_{ij} \frac{\partial^2}{\partial x_i \partial v_j} + mx_i^2 + \kappa \epsilon_{ij} x_i v_j + \frac{\kappa^2}{4m} v_i^2 + 2it. \quad (3.53)
\end{aligned}$$

Using these expressions, one can check explicitly the commutators (3.16), and also that these quadratic generators commute with \hat{S}_1 , \hat{S}_2 and \hat{S}_3 (this also follows from the derivation properties of the commutators and the corresponding commutation of the linear generators \hat{P}_i , \hat{K}_i , and this proves that the Schrödinger equation for the free particle in the noncommutative plane has the Schrödinger algebra as a symmetry. Notice, however, that in this coordinate representation of the non-reduced quantum space the quadratic operators contain second order derivatives, and hence do not generate point transformations for the coordinates x , v . This is in agreement with the results obtained in the reduced space quantization and the Fock space representation. In any case, the fact that the linear generators commute with \hat{S}_1 , \hat{S}_2 and \hat{S}_3 allows to prove that the quadratic ones also commute, and thus generate symmetries of the Schrödinger equation of the free particle in the non-commutative plane.

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References

- [1] H. A. Kastrup, “Gauge properties of the Galilei space,” Nucl. Phys. B **7** (1968) 545.
- [2] R. Jackiw, “Introducing scale symmetry,” Phys. Today **25N1** (1972) 23.
- [3] U. Niederer, “The maximal kinematical invariance group of the free Schrodinger equation.,” Helv. Phys. Acta **45** (1972) 802.
- [4] C. R. Hagen, “Scale and conformal transformations in galilean-covariant field theory,” Phys. Rev. D **5** (1972) 377.
- [5] M. Valenzuela, “Higher-spin symmetries of the free Schrödinger equation,” arXiv:0912.0789 [hep-th].

- [6] X. Bekaert, E. Meunier and S. Moroz, “Symmetries and currents of the ideal and unitary Fermi gases,” JHEP **1202** (2012) 113 [arXiv:1111.3656 [hep-th]].
- [7] J.-M. Lévy-Leblond, *Galilei group and Galilean invariance*. In: *Group Theory and Applications* (E. M. Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972).
- [8] A. Ballesteros, M. Gadella and M. del Olmo, *Moyal quantization of 2+1 dimensional Galilean systems*. *Journ. Math. Phys.* **33** (1992) 3379;
- [9] Y. Brihaye, C. Gónzalez, S. Giller and P. Kosiński, *Galilean invariance in 2 + 1 dimensions*. arXiv:hep-th/9503046.
- [10] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, “Galilean invariant (2+1)-dimensional models with a Chern-Simons-like term and $D = 2$ noncommutative geometry,” *Annals Phys.* **260** (1997) 224 [hep-th/9612017].
- [11] C. Duval and P. A. Horvathy, “Exotic Galilean symmetry in the noncommutative plane, and the Hall effect,” *J. Phys. A* **34** (2001) 10097 [hep-th/0106089].
- [12] P. A. Horvathy and M. S. Plyushchay, “Anyon wave equations and the noncommutative plane,” *Phys. Lett. B* **595** (2004) 547 [hep-th/0404137].
- [13] S. R. Coleman, J. Wess and B. Zumino, “Structure of phenomenological Lagrangians. 1,” *Phys. Rev.* **177** (1969) 2239; C. G. Callan, S. R. Coleman, J. Wess and B. Zumino, “Structure of phenomenological Lagrangians. 2,” *Phys. Rev.* **177** (1969) 2247.
- [14] P. D. Alvarez, J. Gomis, K. Kamimura and M. S. Plyushchay, “(2+1)D Exotic Newton-Hooke Symmetry, Duality and Projective Phase,” *Annals Phys.* **322** (2007) 1556 [hep-th/0702014].
- [15] E. A. Ivanov and V. I. Ogievetsky, “The Inverse Higgs Phenomenon in Nonlinear Realizations,” *Teor. Mat. Fiz.* **25** (1975) 164.
- [16] P. A. Horvathy and M. S. Plyushchay, “Nonrelativistic anyons in external electromagnetic field,” *Nucl. Phys. B* **714** (2005) 269 [hep-th/0502040].
- [17] M. A. del Olmo and M. S. Plyushchay, “Electric Chern-Simons term, enlarged exotic Galilei symmetry and noncommutative plane,” *Annals Phys.* **321** (2006) 2830 [hep-th/0508020].
- [18] J. Gomis and K. Kamimura, “Schrodinger Equations for Higher Order Non-relativistic Particles and N-Galilean Conformal Symmetry,” *Phys. Rev. D* **85**, 045023 (2012) [arXiv:1109.3773 [hep-th]].